Further maths taster work

As well as completing the induction work for single maths, as further mathematicians, you will also need to complete a series of tasks involving two topics that you are yet to encounter in your mathematical journeys. Namely, radians and complex numbers.

Title of Tasks	**Purpose of Task:**	**Time to be taken**	**Outcome Expected**
Task 1: An introduction to radians	To understand that radians are an alternative measurement of angle to degree	1 hour	Notes are created with diagrams and examples.
Task 2: An introduction to complex numbers	Understand the definition of imaginary and complex numbers, manipulate/ work with complex numbers, solve polynomials involving complex numbers	2 hours	Notes are created with diagrams and examples.
Task 3: The Argand diagram	To know how complex numbers are represented on an Argand diagram	1 hour	Notes are created with diagrams and examples
Task 4: Modulus-argument form of a complex number	To know how to write a complex number in modulus-argument form, converting into and out of cartesian form, working with complex numbers in modulus-argument form	2 hours	Notes are created with diagrams and examples.
Task 5: Test your understanding	Answer all questions (and mark your work) from the two exercises at the end of this document	2 hours	All questions answered and marked. Problem questions flagged and to be brought to teacher's attention in September.
Task 5: Maths beyond the classroom	To make it a habit to listen, watch or read around your subject on a field of maths that you find interesting.	On going	A brief discussion/ presentation (does not have to be long) on an area of maths that you find interesting.

Task 1:
Use YouTube to search for videos that explain:

a) what a radian is
b) how to convert between radians and degrees
c) why they are useful

Make notes on these points including any diagrams and examples.

Task 2:
Use YouTube to search for videos that explain:

a) what a complex number is and what an imaginary number is
b) add, subtract, multiply, divide complex numbers
c) the meaning of a complex conjugate
d) how to solve polynomials (quadratics, cubics, quartics) with real and/ or complex roots

Make notes on these points including any diagrams and examples.

Task 3:
Use YouTube to search for videos that explain:

a) who Argand was
b) how to show a complex number on an Argand diagram
c) how to find the modulus and argument of a complex number
d) how to write complex numbers in modulus-argument form and convert between this form and cartesian form

Make notes on these points including any diagrams and examples.

Task 4: Answer the questions from the two exercises located at the end of this document. Mark your work.

Task 5:

If you are studying further maths then that means you are interested in taking maths to a high level at university, whether it be a straight maths course, or a maths related course such as engineering, computing or physics. Whatever course you end up taking, you will need to prove in your application that you are reading beyond classroom texts.

Read/ listen/ watch a book/ podcast/ vodcast of your choosing – whatever interests you, mathematically!

Provide a short summary on your findings – no more than 1 side of A4 – to present back to the class on your return (I do not mind if you want to pair up with another member of the group, as long as you know who is taking further maths next year!)

A book you may want to explore is Fermat's Last Theorem by Simon Singh (ISBN: 978-1841157917)
https://www.youtube.com/watch?v=qiNcEguuFSA

Some YouTube channels that you should definitely subscribe to are (and any more you can find!):

Numberphile
Khan Academy
3blue1brown

You can start with these excellent videos by 3blue1brown on the mathematics behind Covid-19. Exponentials and logarithms form a large chunk of the A-Level course and you will find the topic in pure maths and statistics.

https://www.youtube.com/watch?v=Kas0tIxDvrg

https://www.youtube.com/watch?v=gxAaO2rsdIs

1 Given that $z_1 = 8 - 3\text{i}$ and $z_2 = -2 + 4\text{i}$, find, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a $z_1 + z_2$

b $3z_2$

c $6z_1 - z_2$

2 The equation $z^2 + bz + 14 = 0$, where $b \in \mathbb{R}$ has no real roots.

Find the range of possible values of b. **(3 marks)**

3 The solutions to the quadratic equation $z^2 - 6z + 12 = 0$ are z_1 and z_2.

Find z_1 and z_2, giving each answer in the form $a \pm \text{i}\sqrt{b}$.

4 By using the binomial expansion, or otherwise, show that $(1 + 2\text{i})^5 = 41 - 38\text{i}$. **(3 marks)**

5 $\text{f}(z) = z^2 - 6z + 10$

Show that $z = 3 + \text{i}$ is a solution to $\text{f}(z) = 0$. **(2 marks)**

6 $z_1 = 4 + 2\text{i}$, $z_2 = -3 + \text{i}$

Express, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a z_1^* **b** $z_1 z_2$ **c** $\dfrac{z_1}{z_2}$

7 Write $\dfrac{(7 - 2\text{i})^2}{1 + \text{i}\sqrt{3}}$ in the form $x + \text{i}y$ where $x, y \in \mathbb{R}$.

8 Given that $\dfrac{4 - 7\text{i}}{z} = 3 + \text{i}$, find z in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$. **(2 marks)**

9 $z = \dfrac{1}{2 + \text{i}}$

Express in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

a z^2 **b** $z - \dfrac{1}{z}$

10 Given that $z = a + b\text{i}$, show that $\frac{z}{z^*} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right) + \left(\frac{2ab}{a^2 + b^2}\right)\text{i}$ **(4 marks)**

11 The complex number z is defined by $z = \frac{3 + q\text{i}}{q - 5\text{i}}$, where $q \in \mathbb{R}$.

Given that the real part of z is $\frac{1}{13}$,

a find the possible values of q **(4 marks)**

b write the possible values of z in the form $a + b\text{i}$, where a and b are real constants. **(1 mark)**

12 Given that $z = x + \text{i}y$, find the value of x and the value of y such that $z + 4\text{i}z^* = -3 + 18\text{i}$ where z^* is the complex conjugate of z. **(5 marks)**

13 $z = 9 + 6\text{i}$, $w = 2 - 3\text{i}$

Express $\frac{z}{w}$ in the form $a + b\text{i}$, where a and b are real constants.

14 The complex number z is given by $z = \frac{q + 3\text{i}}{4 + q\text{i}}$ where q is an integer.

Express z in the form $a + b\text{i}$ where a and b are rational and are given in terms of q. **(4 marks)**

15 Given that $6 - 2\text{i}$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(2 marks)**

16 Given that $z = 4 - k\text{i}$ is a root of the equation $z^2 - 2mz + 52 = 0$, where k and m are positive real constants, find the value of k and the value of m. **(4 marks)**

17 $\text{h}(z) = z^3 - 11z + 20$

Given that $2 + \text{i}$ is a root of the equation $\text{h}(z) = 0$, solve $\text{h}(z) = 0$ completely. **(4 marks)**

18 $\text{f}(z) = z^3 + 6z + 20$

Given that $\text{f}(1 + 3\text{i}) = 0$, solve $\text{f}(z) = 0$ completely. **(4 marks)**

19 $\text{f}(z) = z^3 + 3z^2 + kz + 48$, $k \in \mathbb{R}$

Given that $\text{f}(4\text{i}) = 0$,

a find the value of k **(2 marks)**

b find the other two roots of the equation. **(3 marks)**

20 $\text{f}(z) = z^4 - z^3 - 16z^2 - 74z - 60$

a Write $\text{f}(z)$ in the form $(z^2 - 5z - 6)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $\text{f}(z) = 0$. **(3 marks)**

21 $\text{g}(z) = z^4 - 6z^3 + 19z^2 - 36z + 78$

Given that $\text{g}(3 - 2\text{i}) = 0$, find all the roots of $\text{g}(z) = 0$. **(4 marks)**

22 $\text{f}(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

Given that $\text{f}(4) = 0$,

a find the value of p **(1 mark)**

b solve completely the equation $\text{f}(z) = 0$. **(5 marks)**

1 $f(z) = z^2 + 5z + 10$

a Find the roots of the equation $f(z) = 0$, giving your answers in the form $a \pm ib$, where a and b are real numbers. **(3 marks)**

b Show these roots on an Argand diagram. **(1 mark)**

2 $f(z) = z^3 + z^2 + 3z - 5$

Given that $f(-1 + 2i) = 0$,

a find all the solutions to the equation $f(z) = 0$ **(4 marks)**

b show all the roots of $f(z) = 0$ on a single Argand diagram **(2 marks)**

c prove that these three points are the vertices of a right-angled triangle. **(2 marks)**

3 $f(z) = z^4 - z^3 + 13z^2 - 47z + 34$

Given that $z = -1 + 4i$ is a solution to the equation,

a find all the solutions to the equation $f(z) = 0$ **(4 marks)**

b show all the roots on a single Argand diagram. **(2 marks)**

4 The real and imaginary parts of the complex number $z = x + iy$ satisfy the equation

$$(4 - 3i)x - (1 + 6i)y - 3 = 0$$

a Find the value of x and the value of y. **(3 marks)**

b Show z on an Argand diagram. **(1 mark)**

Find the values of:

c $|z|$ **(2 marks)**

d $\arg z$ **(2 marks)**

5 $z_1 = 4 + 2i$, $z_2 = -3 + i$

a Draw points representing z_1 and z_2 on the same Argand diagram. **(1 mark)**

b Find the exact value of $|z_1 - z_2|$. **(2 marks)**

Given that $w = \frac{z_1}{z_2}$,

c express w in the form $a + ib$, where $a, b \in \mathbb{R}$ **(2 marks)**

d find $\arg w$, giving your answer in radians. **(2 marks)**

6 A complex number z is given by $z = a + 4i$ where a is a non-zero real number.

a Find $z^2 + 2z$ in the form $x + iy$, where x and y are real expressions in terms of a. **(4 marks)**

Given that $z^2 + 2z$ is real,

b find the value of a. **(1 mark)**

Using this value for a,

c find the values of the modulus and argument of z, giving the argument in radians and giving your answers correct to 3 significant figures. **(3 marks)**

d Show the complex numbers z, z^2 and $z^2 + 2z$ on a single Argand diagram. **(3 marks)**

7 The complex number z is defined by $z = \dfrac{3 + 5\mathrm{i}}{2 - \mathrm{i}}$

Find:

a $|z|$ **(4 marks)**

b $\arg z$ **(2 marks)**

8 $z = 1 + 2\mathrm{i}$

a Show that $|z^2 - z| = 2\sqrt{5}$. **(4 marks)**

b Find $\arg(z^2 - z)$, giving your answer in radians to 2 decimal places. **(2 marks)**

c Show z and $z^2 - z$ on a single Argand diagram. **(2 marks)**

9 $z = \dfrac{1}{2 + \mathrm{i}}$

a Express in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$,

i z^2 **ii** $z - \dfrac{1}{z}$ **(4 marks)**

b Find $|z^2|$. **(2 marks)**

c Find $\arg\left(z - \frac{1}{z}\right)$, giving your answer in radians to two decimal places. **(2 marks)**

10 $z = \dfrac{a + 3\mathrm{i}}{2 + a\mathrm{i}}$, $a \in \mathbb{R}$

a Given that $a = 4$, find $|z|$.

b Show that there is only one value of a for which $\arg z = \dfrac{\pi}{4}$, and find this value.

11 $z_1 = -1 - \mathrm{i}$, $z_2 = 1 + \mathrm{i}\sqrt{3}$

a Express z_1 and z_2 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leq \pi$. **(2 marks)**

b Find the modulus of:

i $z_1 z_2$ **ii** $\dfrac{z_1}{z_2}$ **(2 marks)**

c Find the argument of:

i $z_1 z_2$ **ii** $\dfrac{z_1}{z_2}$ **(2 marks)**

12 $z = 2 - 2\mathrm{i}\sqrt{3}$

Find:

a $|z|$ **(1 mark)**

b $\arg z$, in terms of π. **(2 marks)**

$w = 4\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

Find:

c $\left|\dfrac{w}{z}\right|$ **(1 mark)**

d $\arg\left(\dfrac{w}{z}\right)$, in terms of π. **(2 marks)**

13 Express $4 - 4\mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leq \pi$, giving r and θ as exact values. **(3 marks)**

Answers 1

1 **a** $6+i$ **b** $-6+12i$ **c** $50-22i$

2 $-2\sqrt{14} < b < 2\sqrt{14}$

3 $3+i\sqrt{3}, 3-i\sqrt{3}$

4 $(1+2i)^5$
$= 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5$
$= 1 + 10i + 40i^2 + 80i^3 + 80i^4 + 32i^5$
$= 1 + 10i - 40 - 80i + 80 + 32i$
$= 41 - 38i$

5 Substitute $z = 3 + i$ into $f(z)$ to get $f(z) = 0$.

6 **a** $4-2i$ **b** $-14-2i$ **c** $-1-i$

7 $\frac{(45-28i)(1-i\sqrt{3})}{(1+\sqrt{3}i)(1-\sqrt{3}i)} = \frac{45-28\sqrt{3}}{4} + \left(\frac{-45\sqrt{3}-28}{4}\right)i$

8 $\frac{4-7i}{3+i} = \frac{(4-7i)(3-i)}{(3+i)(3-i)} = \frac{12-25i+7i^2}{10} = \frac{1}{2} - \frac{5}{2}i$

9 **a** $\frac{3}{25} - \frac{4}{25}i$ **b** $\frac{-8}{5} - \frac{6}{5}i$

10 $\frac{z}{z^*} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{a^2+2abi+b^2i^2}{a^2-b^2i^2}$
$= \frac{a^2-b^2}{a^2+b^2} + \left(\frac{2ab}{a^2+b^2}\right)i$

11 **a** $\frac{3+qi}{q-5i} \times \frac{q+5i}{q+5i} = \frac{3q-5q}{q^2+25} + \frac{q^2+15}{q^2+25}i$
$\frac{-2q}{q^2+25} = \frac{1}{13} \Rightarrow q^2 + 26q + 25 = 0 \Rightarrow q = -1, q = -25$
b $\frac{1}{13} + \frac{8}{13}i, \frac{1}{13} + \frac{64}{65}i$

12 $x + yi + 4i(x - yi) = -3 + 18i$
$(x + 4y) + (4x + y)i = -3 + 18i$
$x + 4y = -3, 4x + y = 18 \Rightarrow x = 5, y = -2$

13 $\frac{(9+6i)(2+3i)}{(2-3i)(2+3i)} = \frac{18+39i+18i^2}{4-9i^2} = 3i$

14 $\frac{(q+3i)(4-qi)}{(4+qi)(4-qi)} = \frac{7q}{q^2+16} + \frac{12-q^2}{q^2+16}i$

15 **a** $6+2i$ **b** $z^2 - 12z + 40$

16 $k = 6, m = 4$

17 $z = 2+i, 2-i$ or -4

18 $z = -2, 1+3i$ or $1-3i$

19 **a** $k = 16$ **b** $-4i$ and -3

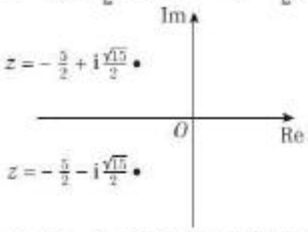
20 **a** $b = 4, c = 10$ **b** $z = 6, -1, -2+\sqrt{6}i$ or $-2-\sqrt{6}i$

21 $3-2i, 3+2i, i\sqrt{6}$ and $-i\sqrt{6}$

22 **a** $p = -18$ **b** $1, 4, \frac{3}{2} + \frac{\sqrt{15}}{2}i$ and $\frac{3}{2} - \frac{\sqrt{15}}{2}i$

Answers 2

1 **a** $z = -\frac{5}{2} + \frac{\sqrt{15}}{2}i$ and $z = -\frac{5}{2} - \frac{\sqrt{15}}{2}i$
b

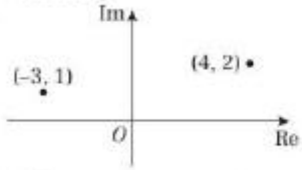


2 **a** $-1+2i, -1-2i$ are two of the roots. These roots can be used to form the quadratic $z^2 + 2z + 5$.
$(z-1)(z^2+2z+5) = f(z)$, so third root is 1.
b Argand diagram showing $-1+2i, -1-2i$ and 1.
c Sides of triangle are $\sqrt{8}, \sqrt{8}$ and 4. $(\sqrt{8})^2 + (\sqrt{8})^2 = 4^2$.

3 **a** $-1+4i, -1-4i, 2, 1$
b Argand diagram showing above roots.

4 **a** $4x - y = 3$
$-3x - 6y = 0 \Rightarrow x = -2y$
$-9y = 3 \Rightarrow y = -\frac{1}{3} \Rightarrow x = \frac{2}{3}$
b Argand diagram showing the point $z = \frac{2}{3} - \frac{1}{3}i$
c $\frac{\sqrt{5}}{3}$
d -0.46 rad

5 **a**

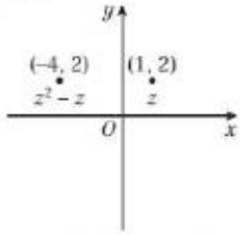


b $5\sqrt{2}$ **c** $-1-i$ **d** $-\frac{3\pi}{4}$

6 **a** $z^2 = (a^2 - 16) + 8ai$
$2z = 2a + 8i$
$z^2 + 2z = (a^2 + 2a - 16) + (8 + 8a)i$
b $a = -1$
c $z = -1 + 4i$
$|z| = \sqrt{17} = 4.12$
$\arg z \approx 1.82$
d Show $z = -1+4i$, $z^2 = -15-8i$ and $z^2 + 2z = -17$ on a single Argand diagram.

7 **a** $z = \frac{(3+5i)(2+i)}{(2-i)(2+i)} = \frac{1}{5} + \frac{13}{5}i$
$|z| = \frac{1}{5}\sqrt{170}$
b $\arg z = 1.49$

8 **a** $z^2 = -3 + 4i$
$z^2 - z = -4 + 2i$
$|-4+2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$
b 2.68
c



9 **a** **i** $\frac{3}{25} - \frac{4}{25}i$ **ii** $\frac{-8}{5} - \frac{6}{5}i$ **b** $\frac{1}{5}$ **c** -2.50

10 **a** $\frac{\sqrt{5}}{2}$
b $\frac{a+3i}{2+ai} = \frac{5a}{4+a^2} + \frac{-a^2+6}{4+a^2}i$,
for $\arg z = \frac{\pi}{4}$ real and imaginary parts must be equal
$\Rightarrow a^2 + 5a - 6 = 0$
$\Rightarrow a = -6$ or 1
a cannot be negative otherwise $\arg z$ is negative
$\therefore a = 1$

11 **a** $z_1 = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$ and
$z_2 = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$
b **i** $2\sqrt{2}$ **ii** $\frac{\sqrt{2}}{2}$
c **i** $-\frac{5\pi}{12}$ **ii** $\frac{11\pi}{12}$

12 **a** $|z| = |2 - 2i\sqrt{3}| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$
b $\arg z = -\frac{\pi}{3}$
c $\left|\frac{w}{z}\right| = 1$
d $\arg\left(\frac{w}{z}\right) = \frac{\pi}{12}$